A SHORT PROOF OF A KNOWN RELATION FOR CONSECUTIVE POWER SUMS

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Abstract. We give a new short proof of the most simple relation between consecutive power sums of the first m positive integers.

1. Introduction

Let

$$S_n(m) = 1^n + 2^n + \ldots + m^n.$$

It is well-known that $S_n(m)$ is a polynomial in m of degree $m+1$. In [1] it was obtained,probably, the most simple relation between $S_n(m)$ and $S_{n-1}(m)$:

$$S_n(m) = m + nS^*_{n-1}(m), \;\; m \geq 1, \;\; n \geq 1 \tag{1}$$

where $S^*_{n-1}(m)$ is obtained by replacing in $S_{n-1}(m)$ the powers m^j by $\frac{m^{j+1}-m}{j+1}$, $j = 0, 1, \ldots, n$. Proof of (1) in [1] was rather long including a complicated induction.

In this note we give quite a different and short proof of (1) with help of the Bernoulli polynomials $B_n(x)$ which are defined by the generating function

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1} \;\; (B_0(x) = 1). \tag{2}$$

There is an almost exhaustive bibliography of the Bernoulli polynomials and numbers at K.Dilcher and I.S.Slavutskii [2].

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2. Proof of Relation

Using (2) for $x = m+1$ and $x = 0$ we have

$$\sum_{n=0}^{\infty}(B_n(m+1) - B_n(0))\frac{t^n}{n!} = \frac{t(e^{(m+1)t}-1)}{e^t - 1}$$

or

$$\sum_{n=0}^{\infty}\left(\frac{B_{n+1}(m+1) - B_{n+1}(0)}{n+1}\right)\frac{t^n}{n!} = \frac{e^{(m+1)t}-1}{e^t-1} = 1 + e^t + \ldots + e^{mt}. \tag{3}$$

On the other hand we have as well

$$\sum_{n=0}^{\infty}(\delta_{n,0} + S_n(m))\frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty}(\delta_{n,0} + 1^n + 2^n + \ldots + m^n)\frac{t^n}{n!} = 1 + e^t + e^{2t} + \ldots + e^{mt}. \tag{4}$$

Therefore, comparing (3) and (4) we conclude that for $n \geq 1$

$$S_n(m) = \frac{B_{n+1}(m+1) - B_{n+1}(0)}{n+1}. \tag{5}$$

Now to prove (1) let us first prove an identity close to (1) for the Bernoulli polynomials

$$B_n^*(x+1) = \frac{B_{n+1}(x+1) - B_{n+1}(0)}{n+1} - x - \delta_{n,0}. \tag{6}$$

Note that

$$(x^j)^* = \frac{x^{j+1} - x}{j+1} = \int_o^x y^j dy - x\int_0^1 y^j dy.$$

Therefore,

$$\sum_{n=0}^{\infty} B_n^*(x+1)\frac{t^n}{n!} = \int_0^x \frac{te^{(y+1)t}}{e^t-1}dy - x\int_0^1 \frac{te^{(y+1)t}}{e^t-1}dy =$$

$$= \frac{e^{(x+1)t} - e^t - x(e^{2t} - e^t)}{e^t - 1} = \frac{e^{(x+1)t} - e^t}{e^t - 1} - xe^t. \tag{7}$$

On the other hand, for the right hand side of (6) we have as in (3)

$$\sum_{n=0}^{\infty}\left(\frac{B_{n+1}(x+1)-B_{n+1}(0)}{n+1}-x-\delta_{n,0}\right)\frac{t^n}{n!}=$$

$$=\frac{e^{(x+1)t}}{e^t-1}-xe^t-1=\frac{e^{(x+1)t}-e^t}{e^t-1}-xe^t. \tag{8}$$

Comparing (7) and (8) we obtain (6).

Now from (5) we have

$$nS_{n-1}(m)=B_n(m+1)-B_n(0).$$

Thus,

$$nS^*_{n-1}(m)=(B_n(m+1)-B_n(0))^*=B^*_n(m+1)$$

and according to (6) and again (5) we find

$$nS^*_{n-1}(m)=S_n(m)-m-\delta_{m,0}$$

and (1) follows ■.

3. Examples

Since $S_1(m)=\frac{m(m+1)}{2}$ then we have

$$S_2(m)=m+\frac{m^3-m}{3}+\frac{m^2-m}{2}=\frac{1}{6}(2m^3+3m^2+m).$$

Furthermore,

$$S_3(m)=m+\frac{1}{2}\left(2\frac{m^4-m}{4}+3\frac{m^3-m}{3}+\frac{m^2-m}{2}\right)=\frac{1}{4}(m^4+2m^3+m^2),$$

$$S_4(m)=m+\frac{m^5-m}{5}+2\frac{m^4-m}{4}+\frac{m^3-m}{3}=\frac{1}{30}(6m^5+15m^4+10m^3-m),$$

$$S_5(m)=m+\frac{1}{6}\left(6\frac{m^6-m}{6}+15\frac{m^5-m}{5}+10\frac{m^4-m}{4}-\frac{m^2-m}{2}\right)=$$

$$= \frac{1}{12}(2m^6 + 6m^5 + 5m^4 - m^2),$$

$$S_6(m) = m + \frac{1}{2}\left(2\frac{m^7 - m}{7} + 6\frac{m^6 - m}{6} + 5\frac{m^5 - m}{5} - \frac{m^3 - m}{3}\right) =$$

$$= \frac{1}{42}(6m^7 + 21m^6 + 21m^5 - 7m^3 + m)$$

etc.

References

[1] 1.V. S. Abramovich, Sums of equiexponent powers of natural numbers, *Kvant* no. 5(1973), 22-25 (in Russian).

[2] 2.K. Dilcher and I. S. Slavutskii, A Bibliography of Bernoulli Numbers, www.mscs.dal.ca/∼ dilcher/bernoulli.html

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